

Brane singularities with mixtures in the bulk

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Abstract

By extending previous analysis of the authors, a systematic study of the singularity structure and possible asymptotic behaviors of five-dimensional braneworld solutions is performed in the case where the bulk is a mixture of an analog of perfect fluid (with a density and pressure depending on the extra coordinate) and a massless scalar field. The two bulk components interact by exchanging energy so that the total energy is conserved. In a particular range of the interaction parameters, we find flat brane general solutions avoiding the singularity at finite distance from the brane, in the same region of the equation of state constant parameter $\gamma = P/\rho$ that we found previously in the absence of the bulk scalar field ($-1 < \gamma < -1/2$).

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1 Introduction

In our previous works [1, 2], we started a systematic study of the singularity structure and possible asymptotic behaviors of five-dimensional braneworld solutions, by parameterizing the bulk field content with an analog of perfect fluid satisfying the equation of state $P = \gamma\rho$, where the ‘pressure’ P and the ‘density’ ρ depend only on the extra dimension Y and γ is a constant parameter. Our motivation was based on the idea of the so-called self-tuning mechanism for the cosmological constant [3, 4], aiming to examine in a model independent way the possibility of avoiding singularities in the bulk at a finite distance from the brane position. We had found three regions of γ leading to qualitatively different behavior:

- The region $\gamma > -1/2$ is very similar to the case of a massless bulk scalar field. Indeed, the existence of a singularity at a finite distance is unavoidable in all solutions with a flat brane, in agreement with earlier works that made similar investigations in different models, using other methods [5, 6]. Moreover, we have shown that the singularity can be avoided (*e.g.* moved at infinite distance) when the brane becomes curved, either positively or negatively. Thus, requiring absence of singularity brings back the cosmological constant problem, since the brane curvature depends on its tension that receives quartic divergent quantum corrections.
- In the region $-1 < \gamma < -1/2$, the curved brane solution becomes singular while the flat brane is regular. Thus, this region seems to avoid the main obstruction of the self-tuning proposal: any value of the brane tension is absorbed in the solution and the brane remains flat.¹
- In the region $\gamma < -1$, corresponding to the analog of a phantom equation of state, the brane can be ripped apart in as much the same way as in a big rip singularity.

¹Of course, the main question is then whether there is a consistent field theory realization of such a fluid producing naturally an effective equation of state of this type [7].

This happens only in the flat case, while curved brane solutions develop ‘standard’ collapse singularities. No regular solution was found in this region.

Moreover, we have shown that all possible singularities at a finite distance from the original position of the brane can be classified in three main classes which we coined collapse type I, collapse type II, and big rip singularities, respectively.

The collapse types I and II met in the asymptotic evolution are both characterized, as their name suggests, by the vanishing of the warp factor. Their differences can be traced in the behavior of the derivative of the warp factor and density of the matter component. In the collapse type I class for instance, the derivative of the warp factor becomes infinite whereas in the collapse type II class it remains finite. The density of the bulk matter, on the other hand, is necessarily divergent asymptotically in the collapse type I class, whereas in the collapse type II class it may approach a constant or even vanish.

An interesting aspect of these two behaviors is that the types of singularity which become asymptotically feasible depend in the first place, on the type of bulk matter: while a massless scalar field, which may be regarded as a fluid with $\gamma = 1$, allows the development of only a collapse type I singularity, a perfect fluid allows in addition the emergence of singularities covering the whole variety of the collapse type II class as well as big rip singularities. The latter are singularities characterized by the divergence of the warp factor, its derivative and the matter density of the fluid, and arise only when the parameter γ is less than -1 . In addition, the possible types are determined by the spatial geometry of the brane: A flat brane allows the development of all the different types of finite-distance singularity whereas a curved brane permits exclusively the formation of collapse type II singularities.

In this paper we extend our previous work by examining the case of a bulk filled with a mixture of fluid and massless scalar field. We let these two bulk entities either interact with each other or simply coexist independently in the bulk. In the latter case, we show that all previous types of singularity are still possible. However, the nature of

each type is now enriched with the behavior of both bulk components. The collapse type I singularity, for example, is characterized by the divergence of the density of either one or even both of the two components, whereas, the big rip singularity is characterized by the divergence only of the density of the fluid while the density of the scalar field vanishes asymptotically. These two types of singularity arise in general solutions. The collapse type II singularity on the other hand, arises in particular solutions and exhibits a possible divergence in the density of the fluid. Apart from these singular solutions, we also find regular ones that lead to avoidance of finite-distance singularities but only for the case of curved branes. In particular, we may avoid finite-distance singularities for a curved brane when $\gamma > -1/2$. In contrast with the previous results, we do not find any range of γ that leads to avoidance of finite-distance singularities for a flat brane. As we mentioned before, the existence of a general regular solution for a flat brane implies that a self-tuning mechanism may be constructed. The failure of our flat-brane model to offer such possibility leads us to its generalization which is implemented by considering an interaction between the two components in the bulk. In this more complicated case we find that for an adequate choice of the interaction parameters and for $-1 < \gamma < -1/2$, the avoidance of singularities is recovered.

Our approach in avoiding finite-distance singularities is to find ranges of the parameter γ and later on of the interaction coupling-coefficients that allow for the existence of solutions that are singular only at infinite distance. We do not, however, consider the possibility of constructing regular solutions with a matching mechanism as in [7]. This mechanism is implemented by exploiting solutions that exhibit a finite-distance singularity that is located only in the half line of the extra dimension away from the position of the space of matching.

The structure of this paper is the following: In Section 2, we start by giving a set up of the basic equations of our model consisting of a brane in a bulk with a scalar field and an analog of perfect fluid. These two bulk components may exchange energy in a way that the total energy is conserved. The field equations are written as a dynamical system

which we analyze with the method of asymptotic splittings, cf. [8]. As a first step in Section 3, we focus on the case in which there is no exchange of energy between the bulk components and derive all possible asymptotic decompositions of the dynamical system together with their dominant balances, i.e., the different possible asymptotic modes of behavior. In particular Subsections 3.4-3.6, are devoted to the asymptotic structure of our braneworlds near finite-distance singularities in the bulk, while in Subsection 3.7, we focus on behavior at infinity. As a second step in Section 4 we consider that the two bulk components interact with each other and resolve the unwanted situation discussed in Subsection 3.7. We conclude and discuss our results in Section 5.

2 Field equations

We consider a braneworld model consisting of a three-brane embedded in a five-dimensional bulk space that is filled with a massless scalar field and an analog of perfect fluid. We assume a bulk metric of the form

$$g_5 = a^2(Y)g_4 + dY^2, \quad (2.1)$$

where g_4 is the four-dimensional flat, de Sitter or anti de Sitter metric, i.e.,

$$g_4 = -dt^2 + f_\kappa^2 g_3, \quad (2.2)$$

where

$$g_3 = dr^2 + h_\kappa^2 g_2 \quad (2.3)$$

and

$$g_2 = d\theta^2 + \sin^2(\theta)d\varphi^2. \quad (2.4)$$

Here $f_\kappa = 1, \cosh(Ht)/H, \cos(Ht)/H$ (H^{-1} is the de Sitter curvature radius) and $h_\kappa = r, \sin r, \sinh r$, respectively. For the scalar field we assume an energy-momentum tensor of the form $T_{AB}^1 = (\rho_1 + P_1)u_A u_B - P_1 g_{AB}$ where $A, B = 1, 2, 3, 4, 5$, $u_A = (0, 0, 0, 0, 1)$ and ρ_1, P_1 is the density and pressure of the scalar field which we take as $P_1 = \rho_1 =$

$\lambda\phi'^2/2$, where the prime denotes differentiation with respect to Y and λ is a parameter. Respectively, the energy-momentum tensor of the fluid is $T_{AB}^2 = (\rho_2 + P_2)u_A u_B - P_2 g_{AB}$ and we assume an equation of state of the form $P_2 = \gamma\rho_2$ between the pressure P_2 and the density ρ_2 with γ being a parameter. All quantities ρ_1 , ρ_2 and P_1 , P_2 are functions of the fifth dimension Y only. The five-dimensional Einstein field equations,

$$G_{AB} = \kappa_5^2 T_{AB}, \quad (2.5)$$

where $\kappa_5^2 = M_5^{-3}$ and M_5 is the five dimensional Planck mass, can then be written as

$$\frac{a''}{a} = -A\lambda\phi'^2 - \frac{2}{3}A(1+2\gamma)\rho_2, \quad (2.6)$$

$$\frac{a'^2}{a^2} = \frac{\lambda A}{3}\phi'^2 + \frac{2A}{3}\rho_2 + \frac{kH^2}{a^2}, \quad (2.7)$$

where $A = \kappa_5^2/4$, $k = 0, \pm 1$ (and the prime ($'$) denotes differentiation with respect to Y). We assume that there is an exchange of energy between the two matter components depending on the values and signs of the two constants ν, σ , such that the total energy is conserved [9], so that we have the following two equations,

$$\lambda\phi'\phi'' + 4\lambda\frac{a'}{a}\phi'^2 = -\frac{\lambda\nu}{2}\frac{a'}{a}\phi'^2 + \sigma\rho_2\frac{a'}{a}, \quad (2.8)$$

$$\rho_2' + 4(\gamma+1)\frac{a'}{a}\rho_2 = \frac{\lambda\nu}{2}\frac{a'}{a}\phi'^2 - \sigma\rho_2\frac{a'}{a}. \quad (2.9)$$

Eqs. (2.6) and (2.7) are not independent, since Eq. (2.6) was derived after substitution of Eq. (2.7) in the field equation $G_{\alpha\alpha} = \kappa_5^2 T_{\alpha\alpha} = 4AT_{\alpha\alpha}$, $\alpha = 1, 2, 3, 4$:

$$\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{kH^2}{a^2} = -\frac{2A}{3}\lambda\phi'^2 - \frac{4A}{3}\gamma\rho_2. \quad (2.10)$$

In our analysis we use the independent Eqs. (2.6), (2.8) and (2.9) to determine the unknown variables a , a' , ϕ' and ρ_2 , while Eq. (2.7) will play the role of a constraint equation for our system.

Our purpose is to find all possible asymptotic behaviors of (general or particular) solutions of the system defined by the dynamical equations (2.6)-(2.9). The most adequate tool for this quest is perhaps the method of asymptotic splittings summarized in

[8]. The first step is to write this system in the form of a suitable dynamical system. We introduce the following set of variables:

$$(x, y, z, w) = (a, a', \phi', \rho_2).$$

The system of equations (2.6), (2.8) and (2.9) then becomes the following dynamical system

$$x' = y \tag{2.11}$$

$$y' = -A\lambda z^2 x - \frac{2}{3}A(1+2\gamma)wx \tag{2.12}$$

$$z' = -\left(4 + \frac{\nu}{2}\right) \frac{yz}{x} + \frac{\sigma}{\lambda} \frac{yw}{xz} \tag{2.13}$$

$$w' = -(4(\gamma+1) + \sigma) \frac{yw}{x} + \frac{\lambda\nu}{2} \frac{yz^2}{x}, \tag{2.14}$$

while equation (2.7) now reads

$$\frac{y^2}{x^2} = \frac{A\lambda}{3} z^2 + \frac{2A}{3} w + \frac{kH^2}{x^2}. \tag{2.15}$$

Since this last equation does not contain derivatives with respect to Y , it is a constraint equation for the system (2.11)-(2.14). The vector field defined by the above system is given by

$$\mathbf{f} = \left(y, -A\lambda z^2 x - \frac{2}{3}A(1+2\gamma)wx, -\left(4 + \frac{\nu}{2}\right) \frac{yz}{x} + \frac{\sigma}{\lambda} \frac{yw}{xz}, -(4(\gamma+1) + \sigma) \frac{yw}{x} + \frac{\lambda\nu}{2} \frac{yz^2}{x} \right)^\top. \tag{2.16}$$

Before we proceed with the analysis of the above system, we introduce the following terminology for the possible singularities to occur at a finite-distance from the brane. Specifically we call a state where:

- i) $a \rightarrow 0, a' \rightarrow \infty, \phi' \rightarrow \infty, \rho_2 \rightarrow 0, \rho_s, \infty$: a singularity of collapse type I,
- ii) $a \rightarrow 0, a' \rightarrow a'_s, \phi' \rightarrow 0, \rho_2 \rightarrow \rho_s, \infty$: a singularity of collapse type II,
- iii) $a \rightarrow \infty, a' \rightarrow -\infty, \phi' \rightarrow 0, \rho_2 \rightarrow \infty$: a big rip singularity,

where a'_s and ρ_s are non-vanishing constants.

In the following Subsections we first analyze the case in which there is no exchange of energy between the two components in the bulk, that is we take $\nu = \sigma = 0$, and later on we examine the very interesting case $\sigma = 0$ and ν arbitrary and we comment on the results of the case $\nu = 0$ and σ arbitrary. The generic case σ, ν nonzero is difficult to study in generality and classify all possible behaviors.

3 Non-interacting mixture in the bulk

In this Section we let $\nu = \sigma = 0$ so that the system Eqs. (2.11)-(2.14) becomes

$$x' = y \quad (3.1)$$

$$y' = -A\lambda z^2 x - \frac{2}{3}A(1+2\gamma)wx \quad (3.2)$$

$$z' = -4\frac{yz}{x} \quad (3.3)$$

$$w' = -4(\gamma+1)\frac{yw}{x}, \quad (3.4)$$

while equation (2.15) remains the same. The vector field of the above system is

$$\mathbf{f} = \left(y, -A\lambda z^2 x - \frac{2}{3}A(1+2\gamma)wx, -4\frac{yz}{x}, -4(\gamma+1)\frac{yw}{x} \right)^\top, \quad (3.5)$$

and there are three possible ways of decomposing it. We analyze them in turn in the following Subsections.

3.1 Decomposition I

The first way of decomposing the vector field (3.5) is to assume that its dominant part is given by the form

$$\mathbf{f}^{(0)} = \left(y, -A\lambda z^2 x, -4\frac{yz}{x}, -4(\gamma+1)\frac{yw}{x} \right)^\top, \quad (3.6)$$

while its candidate subdominant part is:

$$\mathbf{f}^{(1)} = \left(0, -\frac{2}{3}A(1+2\gamma)wx, 0, 0 \right)^\top. \quad (3.7)$$

At this point we wish to determine all *dominant balances*, that is pairs of the form

$$\mathcal{B} = \{\mathbf{a}, \mathbf{p}\}, \quad \text{where} \quad \mathbf{a} = (\alpha, \beta, c, \zeta), \quad \mathbf{p} = (p, q, r, s), \quad (3.8)$$

with

$$(p, q, r, s) \in \mathbb{Q}^4 \quad \text{and} \quad (\alpha, \beta, c, \zeta) \in \mathbb{C}^4 \setminus \{\mathbf{0}\}, \quad (3.9)$$

that describe all possible asymptotic behaviors around the assumed position of the singularity at Y_s . We thus insert

$$(x, y, z, w) = (\alpha \Upsilon^p, \beta \Upsilon^q, c \Upsilon^r, \zeta \Upsilon^s), \quad (3.10)$$

where $\Upsilon = Y - Y_s$, into the asymptotic system defined by the first decomposition, that is

$$x' = y \quad (3.11)$$

$$y' = -A\lambda z^2 x \quad (3.12)$$

$$z' = -4 \frac{yz}{x} \quad (3.13)$$

$$w' = -4(\gamma + 1) \frac{yw}{x}. \quad (3.14)$$

This leads us to the list of all possible dominant balances. For each balance we need to check that the *dominance condition*,

$$\lim_{\Upsilon \rightarrow 0} \frac{\mathbf{f}^{(1)}(\mathbf{a}\Upsilon^{\mathbf{p}})}{\Upsilon^{\mathbf{p}-1}} = 0, \quad (3.15)$$

is satisfied and then discard those balances that do not satisfy Eq. (3.15). We end up with the following acceptable balances²:

$${}_I\mathcal{B}_1 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), \zeta), (1/4, -3/4, -1, -(\gamma+1))\}, \quad \gamma < 1, \quad (3.16)$$

$${}_I\mathcal{B}_2 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), 0), (1/4, -3/4, -1, s)\}, \quad (3.17)$$

$${}_I\mathcal{B}_3 = \{(\alpha, \alpha, 0, \zeta), (1, 0, -1, -4(\gamma+1))\}, \quad \gamma \leq -1/2, \quad (3.18)$$

$${}_I\mathcal{B}_4 = \{(\alpha, \alpha, 0, 0), (1, 0, -1, s)\}, \quad (3.19)$$

$${}_I\mathcal{B}_5 = \{(\alpha, 0, 0, 0), (0, -1, -1, s)\}. \quad (3.20)$$

The balance ${}_I\mathcal{B}_5$ leads to ρ_2 being identically zero which means that it describes a behavior that applies in the case of bulk filled exclusively with the scalar field. We studied this balance in our previous work in [2] and found it to be unacceptable. We therefore do not consider it any further in this paper.

The balances ${}_I\mathcal{B}_{1-4}$ are exact solutions of the system (3.11)-(3.14). The constraint Eq. (2.15) has contributed in the Eq. (3.12) with all its terms excluding the term of the fluid density. We can therefore substitute these balances in the constraint equation (2.15) neglecting the term of the fluid density and find out if they correspond to a flat or curved brane. We find that for $\gamma \neq -1/2$, the balances ${}_I\mathcal{B}_1$ and ${}_I\mathcal{B}_2$ correspond to a flat brane while the balances ${}_I\mathcal{B}_3$ and ${}_I\mathcal{B}_4$ correspond to a curved brane with the arbitrary constant α satisfying $\alpha^2 = kH^2$.

The value $\gamma = -1/2$ is of special interest for our analysis since for this value of γ the system (3.1)-(3.4) identifies with the system of this first decomposition (3.11)-(3.14). This means that the balances we find for $\gamma = -1/2$ are exact solutions of the system (3.1)-(3.4) and they should therefore satisfy the entire constraint equation (2.15). These balances are: ${}_I\mathcal{B}_2$ (flat brane), ${}_I\mathcal{B}_4$ (curved brane with $\alpha^2 = kH^2$), ${}_I\mathcal{B}_3$ (flat/curved brane with $\zeta = 3/(2A)(1 - kH^2/\alpha^2)$ and ${}_I\mathcal{B}_1$ (curved brane with $\zeta = -3/(2A)(kH^2/\alpha^2)$).

²In the balance ${}_I\mathcal{B}_1$ the coefficient $\sqrt{3}/(4\sqrt{A\lambda})$ may also be $-\sqrt{3}/(4\sqrt{A\lambda})$. This is true for every balance we find that has a square root in the coefficient of ϕ' , but for simplicity we examine these balances only for the (+) sign.

3.2 Decomposition II

The second way of decomposing the vector field (3.5) is to take its dominant part to be

$$\mathbf{f}^{(0)} = \left(y, -\frac{2}{3}A(1+2\gamma)wx, -4\frac{yz}{x}, -4(\gamma+1)\frac{yw}{x} \right)^\top. \quad (3.21)$$

Its candidate subdominant part reads,

$$\mathbf{f}^{(1)} = (0, -A\lambda z^2 x, 0, 0)^\top. \quad (3.22)$$

For this second decomposition the system is given by the following equations

$$x' = y \quad (3.23)$$

$$y' = -\frac{2}{3}A(1+2\gamma)wx \quad (3.24)$$

$$z' = -4\frac{yz}{x} \quad (3.25)$$

$$w' = -4(\gamma+1)\frac{yw}{x}, \quad (3.26)$$

and the acceptable balances are calculated to be³

$${}_{II}\mathcal{B}_1 = \{(\alpha, \alpha p, c, 3p^2/(2A)), (p, p-1, -4p, -2)\}, \quad |\gamma| > 1, \quad (3.27)$$

$${}_{II}\mathcal{B}_2 = \{(\alpha, \alpha p, 0, 3p^2/(2A)), (p, p-1, r, -2)\}, \quad \gamma \neq -1, -1/2, \quad (3.28)$$

$${}_{II}\mathcal{B}_3 = \{(\alpha, \alpha, 0, 0), (1, 0, r, -2)\}, \quad \gamma \neq -1/2, \quad (3.29)$$

$${}_{II}\mathcal{B}_4 = \{(\alpha, 0, 0, 0), (0, -1, r, -2)\}, \quad \gamma \neq -1/2, \quad (3.30)$$

where $p = 1/(2(\gamma+1))$. Following the same trend as we did for Decomposition I, we see that the constraint Eq. (2.15) has contributed in the Eq. (3.24) with all its terms excluding the term of the derivative of the scalar field. We therefore substitute the balances ${}_{II}\mathcal{B}_{1-3}$ in the constraint equation Eq. (2.15) neglecting the term of the derivative of the scalar field and find that the balances ${}_{II}\mathcal{B}_1$ and ${}_{II}\mathcal{B}_2$ correspond to a flat brane while the balance ${}_{II}\mathcal{B}_3$ corresponds to a curved brane with $\alpha^2 = kH^2$.

³The balance ${}_{II}\mathcal{B}_4$ is not analyzed any further since it leads to ϕ' being identically zero and a similar argument applies as in the case of ${}_I\mathcal{B}_5$ discussed above, in the decomposition I.

3.3 Decomposition III

The third way of decomposing the vector field (3.5) is to assume that all terms are dominant so that the system is given by Eqs. (3.1)-(3.4). For this third decomposition the dominant balances are

$$_{III}\mathcal{B}_1 = \{(\alpha, \alpha/4, c, 3/(32A) - \lambda c^2/2), (1/4, -3/4, -1, -2)\}, \quad \gamma = 1, \quad (3.31)$$

$$_{III}\mathcal{B}_2 = \{(\alpha, \alpha p, 0, 3p^2/(2A)), (p, p-1, -1, -2)\}, \quad \gamma \neq -1, -1/2, \quad (3.32)$$

$$_{III}\mathcal{B}_3 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), 0), (1/4, -3/4, -1, -2)\}, \quad (3.33)$$

$$_{III}\mathcal{B}_4 = \{(\alpha, \alpha, 0, 0), (1, 0, -1, -2)\}, \quad (3.34)$$

with $p = 1/(2(\gamma + 1))$. These balances are exact solutions of the system (3.1)-(3.4) and they should therefore satisfy the constraint equation (2.15). We find that the balances $_{III}\mathcal{B}_1$, $_{III}\mathcal{B}_2$ and $_{III}\mathcal{B}_3$ correspond to a flat brane, while the balance $_{III}\mathcal{B}_4$ corresponds to a curved brane with $\alpha^2 = kH^2$. Notice that the three decompositions considered above exhaust all possible asymptotic ways that the vector field (3.5) can split.

Subsections 3.4-3.7 are the heart of this Section that focuses on non-interacting bulk components. We have grouped the possible balances $_{I-III}\mathcal{B}$ found above into four different sets according to the type of singularity they lead to, or, their regular behavior. For each particular balance we follow the method of asymptotic splittings up to the point where we end up with a well-defined series expansion. These expansions serve to completely justify our claims that the asymptotic behavior of the braneworld is the one claimed. We find that in all cases these behaviors result from Puiseux representations, in particular there are no logarithmic terms present in any of the expansions.

3.4 Collapse type I singularities

In this Section, we analyze the balances $_I\mathcal{B}_1$, $_{III}\mathcal{B}_3$, $_I\mathcal{B}_2$, $_{II}\mathcal{B}_2$, $_{III}\mathcal{B}_1$ and $_{III}\mathcal{B}_2$ that describe the asymptotics around collapse type I singularities.

3.4.1 The balance ${}_I\mathcal{B}_1$

We start with the analysis of the balance ${}_I\mathcal{B}_1$ that corresponds to a flat brane for $\gamma \neq -1/2$. We will show that for different values of γ this balance implies different behaviors of the matter density of the fluid around a collapse type I singularity. We first have to calculate for this balance the \mathcal{K} -matrix given by

$${}_I\mathcal{K}_1 = D\mathbf{f}^{(0)}(\mathbf{a}) - \text{diag } \mathbf{p}, \quad (3.35)$$

where $D\mathbf{f}^{(0)}(\mathbf{a})$ is the Jacobian matrix of the dominant part $\mathbf{f}^{(0)}$ in Eq. (3.6),

$$D\mathbf{f}^{(0)}(x, y, z, w) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda Az^2 & 0 & -2\lambda Azx & 0 \\ 4\frac{yz}{x^2} & -4\frac{z}{x} & -4\frac{y}{x} & 0 \\ 4(\gamma+1)\frac{yw}{x^2} & -4(1+\gamma)\frac{w}{x} & 0 & -4(1+\gamma)\frac{y}{x} \end{pmatrix}, \quad (3.36)$$

evaluated on \mathbf{a} . We have that $\mathbf{a} = (\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), \zeta)$ and $\mathbf{p} = (1/4, -3/4, -1, -(\gamma+1))$, so that the \mathcal{K} -matrix in this case is

$${}_I\mathcal{K}_1 = D\mathbf{f}^{(0)}((\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), \zeta)) - \text{diag}(1/4, -3/4, -1, -(\gamma+1)) =$$

$$= \begin{pmatrix} -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & \frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & 0 & 0 \\ (1+\gamma)\frac{\zeta}{\alpha} & -4(1+\gamma)\frac{\zeta}{\alpha} & 0 & 0 \end{pmatrix}.$$

Next, we calculate the \mathcal{K} -exponents for this balance. These exponents are the eigenvalues of the matrix ${}_I\mathcal{K}_1$ and constitute its spectrum, $\text{spec}({}_I\mathcal{K}_1)$. We wish to build series expansions of the variables in the form

$$\mathbf{x} = \Upsilon^{\mathbf{p}}(\mathbf{a} + \sum_{j=1}^{\infty} \mathbf{c}_j \Upsilon^{j/S}), \quad (3.37)$$

where $\mathbf{x} = (x, y, z, w)$, $\mathbf{c}_j = (c_{j1}, c_{j2}, c_{j3}, c_{j4})$, and S is the least common multiple of the denominators of the positive \mathcal{K} -exponents and the non-dominant exponents $q^{(1)}$ defined by the requirement

$$\frac{\mathbf{f}^{(1)}(\Upsilon^{\mathbf{p}})}{\Upsilon^{\mathbf{p}-1}} \sim \Upsilon^{q^{(1)}}, \quad (3.38)$$

(cf. [8], [10]). The arbitrary constants of any particular or general solution first appear in those terms in the series (3.37) whose coefficients \mathbf{c}_k have indices $k = \varrho S$, where ϱ is a non-negative \mathcal{K} -exponent. The number of non-negative \mathcal{K} -exponents therefore equals the number of arbitrary constants that appear in the series expansions of (3.37). There is always the -1 exponent that corresponds to an arbitrary constant that is the position of the singularity, Y_s .

The balance ${}_I\mathcal{B}_1$ corresponds thus to a general solution in our case if and only if it possesses three non-negative \mathcal{K} -exponents (the fourth arbitrary constant is the position of the singularity, Y_s). Actually, once we use the constraint equation (2.15), one of the three arbitrary constants corresponding to the three non-negative \mathcal{K} -exponents will be set to a specific value, so that the general solution of the dynamical system (3.1)-(3.4) with constraint equation (2.15) will exhibit three in total arbitrary constants taken into account also the singularity position Y_s . Here we find

$$\text{spec}({}_I\mathcal{K}_1) = \{-1, 0, 0, 3/2\}. \quad (3.39)$$

The double multiplicity of the zero \mathcal{K} -exponent reflects the fact that there are two arbitrary constants in this dominant balance. In total we have three non-negative \mathcal{K} -exponents which means that this balance indeed corresponds to a general solution.

Naturally, the behavior of ρ_2 depends on the exponent $-(\gamma + 1)$. We try inserting different values of γ so that to trace all possible asymptotics for ρ_2 keeping in mind that

$\gamma \leq 1$. For instance, for $\gamma = 0$ we have that

$${}_I\mathcal{B}_1 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), \zeta), (1/4, -3/4, -1, -1)\} \quad (3.40)$$

and substituting in the system (3.1)-(3.4) the particular value $\gamma = 0$ and the forms

$$x = \sum_{j=0}^{\infty} c_{j1} \Upsilon^{j/2+1/4}, \quad y = \sum_{j=0}^{\infty} c_{j2} \Upsilon^{j/2-3/4}, \quad z = \sum_{j=0}^{\infty} c_{j3} \Upsilon^{j/2-1}, \quad w = \sum_{j=0}^{\infty} c_{j4} \Upsilon^{j/2-1},$$

we arrive at the following asymptotic expansions:

$$x = \alpha \Upsilon^{1/4} + \alpha \zeta / 3 \Upsilon^{5/4} + c_{31} \Upsilon^{7/4} + \dots, \quad (3.41)$$

$$y = \alpha / 4 \Upsilon^{-3/4} + 5\alpha \zeta / 12 \Upsilon^{1/4} + 7/4 c_{31} \Upsilon^{3/4} + \dots, \quad (3.42)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda}) \Upsilon^{-1} - \zeta \sqrt{3}/(3\sqrt{A\lambda}) - c_{31} \sqrt{3}/(\alpha \sqrt{A\lambda}) \Upsilon^{1/2} + \dots, \quad (3.43)$$

$$w = \zeta \Upsilon^{-1} - 4\zeta^2/3 - 4\zeta c_{31}/\alpha \Upsilon^{1/2} + \dots. \quad (3.44)$$

The next step in our analysis is to check if for each j satisfying $j/2 = \varrho$ with ϱ a positive eigenvalue, the corresponding eigenvector v of the transpose of the ${}_I\mathcal{K}_1$ matrix is such that the compatibility conditions hold, namely,

$$v^\top \cdot P_j = 0, \quad (3.45)$$

where P_j are polynomials in $\mathbf{c}_1, \dots, \mathbf{c}_{j-1}$ given by

$$({}_I\mathcal{K}_1 - (j/2)\mathcal{I})\mathbf{c}_j = P_j. \quad (3.46)$$

Here the corresponding relation $j/2 = 3/2$, is valid only for $j = 3$ and the compatibility

condition indeed holds since,

$$({}_I\mathcal{K}_1 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ \frac{\zeta}{\alpha} & -4\frac{\zeta}{\alpha} & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} c_{31} \\ \frac{7}{4}c_{31} \\ -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}}c_{31} \\ -\frac{4\zeta}{\alpha}c_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.47)$$

This means that a representation of the solution asymptotically with a Puiseux series as this is given by Eqs. (3.41)-(3.44) is valid. We thus conclude that as $Y \rightarrow Y_s$, or equivalently as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.48)$$

We consider also the case $\gamma = -1/2$ for which we find

$$x = \alpha\Upsilon^{1/4} - \alpha\sqrt{A\lambda}/\sqrt{3}c_{33}\Upsilon^{7/4} + \dots, \quad (3.49)$$

$$y = \alpha/4\Upsilon^{-3/4} - 7\alpha\sqrt{A\lambda}/(4\sqrt{3})c_{33}\Upsilon^{3/4} + \dots, \quad (3.50)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda})\Upsilon^{-1} + c_{33}\Upsilon^{1/2} + \dots, \quad (3.51)$$

$$w = \zeta\Upsilon^{-1/2} + 2\zeta\sqrt{A\lambda}/\sqrt{3}c_{33}\Upsilon + \dots, \quad (3.52)$$

where $\zeta = -3kH^2/(2A\alpha^2)$. The compatibility condition for $j = 3$ is satisfied in the

following way:

$$({}_I\mathcal{K}_1 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ \frac{\zeta}{2\alpha} & -2\frac{\zeta}{\alpha} & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ -\frac{7}{4}\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ c_{33} \\ 2\zeta\frac{\sqrt{A\lambda}}{\sqrt{3}}c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.53)$$

We therefore see that for $\gamma = -1/2$ Eqs (3.49)-(3.52) express the asymptotic behavior that corresponds to a curved brane around a collapse I singularity with the density of the fluid diverging, i.e as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.54)$$

For $\gamma = -2$ we find a different behavior

$$x = \alpha\Upsilon^{1/4} - \sqrt{A\lambda}\alpha/\sqrt{3}c_{33}\Upsilon^{7/4} + \dots, \quad (3.55)$$

$$y = \alpha/4\Upsilon^{-3/4} - 7\sqrt{A\lambda}\alpha/(4\sqrt{3})c_{33}\Upsilon^{3/4} + \dots, \quad (3.56)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda})\Upsilon^{-1} + c_{33}\Upsilon^{1/2} + \dots, \quad (3.57)$$

$$w = \zeta\Upsilon - 4\zeta\sqrt{A\lambda}/\sqrt{3}c_{33}\Upsilon^{5/2} + \dots \quad (3.58)$$

We ought to check the compatibility condition for $j = 3$. We find that this is trivially

satisfied since

$$({}_I\mathcal{K}_1 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ -\frac{\zeta}{\alpha} & 4\frac{\zeta}{\alpha} & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ -\frac{7}{4}\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ c_{33} \\ -4\zeta\frac{\sqrt{A\lambda}}{\sqrt{3}}c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.59)$$

The relations (3.55)-(3.58) are therefore valid representations of a general solution around the singularity at Y_s . We can therefore conclude that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow 0. \quad (3.60)$$

A yet different behavior is met for $\gamma = -1$:

$$x = \alpha\Upsilon^{1/4} - \sqrt{A\lambda}\alpha/\sqrt{3}c_{33}\Upsilon^{7/4} + \dots, \quad (3.61)$$

$$y = \alpha/4\Upsilon^{-3/4} - 7\sqrt{A\lambda}\alpha/(4\sqrt{3})c_{33}\Upsilon^{3/4} + \dots, \quad (3.62)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda})\Upsilon^{-1} + c_{33}\Upsilon^{1/2} + \dots, \quad (3.63)$$

$$w = \zeta + \dots \quad (3.64)$$

The compatibility condition at $j = 3$ holds true since we find

$$({}_I\mathcal{K}_1 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ -\frac{7}{4}\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ c_{33} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.65)$$

so that the relations (3.61)-(3.64) affirm that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \zeta. \quad (3.66)$$

3.4.2 The balance ${}_{III}\mathcal{B}_3$

We now move on to examine the next balance ${}_{III}\mathcal{B}_3 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), 0), (1/4, -3/4, -1, -2)\}$ which corresponds to a potentially general solution of a flat brane. The \mathcal{K} -matrix for this balance is

$${}_{III}\mathcal{K}_3 = D\mathbf{f}(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda}), 0) - \text{diag}(1/4, -3/4, -1, -2), \quad (3.67)$$

where $D\mathbf{f}$ is the Jacobian matrix of the vector field \mathbf{f} in Eq. (3.5). The eigenvalues of the ${}_{III}\mathcal{K}_3$ matrix are

$$\text{spec}({}_{III}\mathcal{K}_3) = \{-1, 0, 3/2, 1 - \gamma\}. \quad (3.68)$$

Setting $\gamma = 0$ we get

$$\text{spec}({}_{III}\mathcal{K}_3) = \{-1, 0, 3/2, 1\}. \quad (3.69)$$

We therefore have three non-negative \mathcal{K} -exponents which means that in this case the balance indeed corresponds to a general solution. The asymptotic expansions of the

variables in this case are

$$x = \alpha \Upsilon^{1/4} + 2/3 A \alpha c_{24} \Upsilon^{5/4} - \alpha \sqrt{A\lambda/3} c_{33} \Upsilon^{7/4} + \dots, \quad (3.70)$$

$$y = \alpha/4 \Upsilon^{-3/4} + 5/6 A \alpha c_{24} \Upsilon^{1/4} - 7/4 \alpha \sqrt{A\lambda/3} c_{33} \Upsilon^{3/4} + \dots \quad (3.71)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda}) \Upsilon^{-1} - 2\sqrt{A/(3\lambda)} c_{24} + c_{33} \Upsilon^{1/2} + \dots, \quad (3.72)$$

$$w = c_{24} \Upsilon^{-1} + \dots \quad (3.73)$$

We have to check the compatibility conditions for $j = 2$ and $j = 3$. Since we find

$$({}_{III}\mathcal{K}_3 - \mathcal{I})\mathbf{c}_2 = \begin{pmatrix} -\frac{5}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{1}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & -(2/3)A\alpha \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2A\alpha}{3}c_{24} \\ \frac{5A\alpha}{6}c_{24} \\ -\frac{2\sqrt{A}}{\sqrt{3\lambda}}c_{24} \\ c_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.74)$$

and

$$({}_{III}\mathcal{K}_3 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & -(2/3)A\alpha \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ -\frac{7}{4}\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ c_{33} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.75)$$

the compatibility conditions for $j = 2$ and $j = 3$ hold true. Eqs. (3.70)-(3.73) then imply that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.76)$$

In order to be able to compare and contrast our results found in this Subsection with the ones that are presented later in Subsection 3.7, we find it necessary to analyze here two more values of γ , namely $\gamma = -1/2$ and $\gamma = -3/4$. For $\gamma = -1/2$ we find that the eigenvalues of the $_{III}\mathcal{K}_3$ matrix read

$$\text{spec}(_{III}\mathcal{K}_3) = \{-1, 0, 3/2, 3/2\}. \quad (3.77)$$

For this value of γ we find the asymptotic behavior

$$x = \alpha \Upsilon^{1/4} - \sqrt{A\lambda}/\sqrt{3} \alpha c_{33} \Upsilon^{7/4} + \dots, \quad (3.78)$$

$$y = \alpha/4 \Upsilon^{-3/4} - 7\sqrt{A\lambda}/(4\sqrt{3}) \alpha c_{33} \Upsilon^{3/4} + \dots, \quad (3.79)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda}) \Upsilon^{-1} + c_{33} \Upsilon^{1/2} + \dots, \quad (3.80)$$

$$w = c_{34} \Upsilon^{-1/2} + \dots. \quad (3.81)$$

Since

$$(_{III}\mathcal{K}_3 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}} c_{33} \\ -\frac{7\sqrt{A\lambda}\alpha}{4\sqrt{3}} c_{33} \\ c_{33} \\ c_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.82)$$

the compatibility condition for $j = 3$ is trivially satisfied, and as it follows from Eqs. (3.78)-(3.81) as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.83)$$

Finally, for $\gamma = -3/4$ the eigenvalues become

$$\text{spec}(_{III}\mathcal{K}_3) = \{-1, 0, 3/2, 7/4\}, \quad (3.84)$$

while the asymptotic behavior now is

$$x = \alpha \Upsilon^{1/4} - \sqrt{A\lambda}/\sqrt{3}\alpha c_{63} \Upsilon^{7/4} + 16/33A\alpha c_{74} \Upsilon^2 + \dots, \quad (3.85)$$

$$y = \alpha/4 \Upsilon^{-3/4} - 7\sqrt{A\lambda}/(4\sqrt{3})\alpha c_{63} \Upsilon^{3/4} + 32/33A\alpha c_{74} \Upsilon + \dots \quad (3.86)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda}) \Upsilon^{-1} + c_{63} \Upsilon^{1/2} - 16\sqrt{3A}/(33\sqrt{\lambda}) c_{74} \Upsilon^{3/4} + \dots, \quad (3.87)$$

$$w = c_{74} \Upsilon^{-1/4} + \dots \quad (3.88)$$

We check the compatibility conditions for $j = 6$ (note that here $S = 4$ and we see that for the eigenvalue $\varrho = 3/2$ the corresponding arbitrary constant appears when $j = \varrho S = 6$) and for $j = 7$ are again trivially satisfied since we find that:

$$({}_{III}\mathcal{K}_3 - 3/2\mathcal{I})\mathbf{c}_6 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 1/3A\alpha \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}}{\sqrt{3}}\alpha c_{63} \\ -\frac{7\sqrt{A\lambda}}{4\sqrt{3}}\alpha c_{63} \\ c_{63} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.89)$$

and

$$({}_{III}\mathcal{K}_3 - 7/4\mathcal{I})\mathbf{c}_7 = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -\frac{3}{16} & -1 & -\frac{\alpha\sqrt{3A\lambda}}{2} & 1/3A\alpha \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{16}{33}A\alpha c_{74} \\ \frac{32}{33}A\alpha c_{74} \\ -\frac{16}{33}\sqrt{3A/\lambda}c_{74} \\ c_{74} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.90)$$

Hence Eqs. (3.85)-(3.88) show that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.91)$$

3.4.3 The balance ${}_I\mathcal{B}_2$

Here we consider the balance ${}_I\mathcal{B}_2$ that corresponds to a flat brane. The eigenvalues of the \mathcal{K} -matrix for this balance are

$$\text{spec}({}_I\mathcal{K}_2) = \{-1, 0, 3/2, -1 - \gamma - s\}. \quad (3.92)$$

For $s = 1/2$ and $\gamma = -5/2$ we get

$$\text{spec}({}_I\mathcal{K}_2) = \{-1, 0, 3/2, 1\}. \quad (3.93)$$

We then find the following expansions

$$x = \alpha\Upsilon^{1/4} - \sqrt{A\lambda/3}\alpha c_{33}\Upsilon^{7/4} + \dots \quad (3.94)$$

$$y = \alpha/4\Upsilon^{-3/4} - 7/4\sqrt{A\lambda/3}\alpha c_{33}\Upsilon^{3/4} + \dots \quad (3.95)$$

$$z = \sqrt{3}/(4\sqrt{A\lambda})\Upsilon^{-1} + c_{33}\Upsilon^{1/2} + \dots \quad (3.96)$$

$$w = c_{24}\Upsilon^{3/2} + \dots \quad (3.97)$$

We have to check the compatibility conditions for $j = 2$ and $j = 3$. Since

$$({}_I\mathcal{K}_2 - \mathcal{I})\mathbf{c}_2 = \begin{pmatrix} -\frac{5}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{1}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.98)$$

and

$$({}_I\mathcal{K}_2 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -\frac{\alpha\sqrt{3A\lambda}}{2} & 0 \\ \frac{\sqrt{3}}{4\alpha\sqrt{A\lambda}} & -\frac{\sqrt{3}}{\alpha\sqrt{A\lambda}} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ -\frac{7}{4}\frac{\sqrt{A\lambda}\alpha}{\sqrt{3}}c_{33} \\ c_{33} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.99)$$

the compatibility conditions are indeed satisfied and from Eqs. (3.94)-(3.97) we see that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow 0. \quad (3.100)$$

3.4.4 The balance ${}_{III}\mathcal{B}_1$

The balance ${}_{III}\mathcal{B}_1$ that corresponds to a potentially general solution of a flat brane is valid only for $\gamma = 1$ and describes the asymptotics around a collapse type I singularity. Here the eigenvalues of the matrix ${}_{III}\mathcal{K}_1$ are

$$\text{spec}({}_{III}\mathcal{K}_1) = \{-1, 0, 0, 3/2\}. \quad (3.101)$$

Since we find three non-negative \mathcal{K} -exponents for this balance we conclude that it indeed corresponds to a general solution.

Here we find the following expressions

$$x = \alpha\Upsilon^{1/4} - \alpha/(4c)c_{33}\Upsilon^{7/4} + \dots \quad (3.102)$$

$$y = \alpha/4\Upsilon^{-3/4} - 7\alpha/(16c)c_{33}\Upsilon^{3/4} + \dots \quad (3.103)$$

$$z = c\Upsilon^{-1} + c_{33}\Upsilon^{1/2} + \dots \quad (3.104)$$

$$w = (3/(32A) - \lambda c^2/2)\Upsilon^{-2} + (2/c)(3/(32A) - \lambda c^2/2)c_{33}\Upsilon^{-1/2} + \dots \quad (3.105)$$

and the compatibility condition for $j = 3$ is valid since

$$({}_{III}\mathcal{K}_1 - 3/2\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -\frac{7}{4} & 1 & 0 & 0 \\ -\frac{3}{16} & -\frac{3}{4} & -2A\lambda c\alpha & -2A\alpha \\ \frac{c}{\alpha} & -4\frac{c}{\alpha} & -\frac{3}{2} & 0 \\ \frac{3}{16\alpha A} - \frac{\lambda c^2}{\alpha} & -\frac{3}{4\alpha A} + \frac{4\lambda c^2}{\alpha} & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} -\frac{\alpha}{4c}c_{33} \\ -\frac{7\alpha}{16c}c_{33} \\ c_{33} \\ \frac{2}{c}\zeta c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.106)$$

where $\zeta = 3/(32A) - \lambda c^2/2$. We thus conclude that as $\Upsilon \rightarrow 0$, Eqs. (3.102)-(3.105) describe the asymptotics around a collapse I singularity with the density of the fluid diverging there, i.e.,

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.107)$$

3.4.5 The balance ${}_{III}\mathcal{B}_2$

Finally, the balance ${}_{III}\mathcal{B}_2$ corresponds to a potentially general solution of a flat brane and has the following \mathcal{K} -exponents

$$\text{spec}({}_{III}\mathcal{K}_2) = \{-1, 0, -2(p-1), 1-4p\}. \quad (3.108)$$

The last two \mathcal{K} -exponents are positive when $\gamma < -1$ or $\gamma > 1$. We consider here the case $\gamma > 1$ because as we show below it leads to the emergence of a collapse I singularity while the case $\gamma < -1$ implies the existence of a big rip singularity and it is considered later in Subsection 3.6. We set $\gamma = 2$. Then $p = 1/6$ for which the balance and the \mathcal{K} -exponents read

$${}_{III}\mathcal{B}_2 = \{(\alpha, \alpha/6, 0, 1/(24A)), (1/6, -5/6, -1, -2)\} \quad (3.109)$$

and

$$\text{spec}({}_{III}\mathcal{K}_2) = \{-1, 0, 5/3, 1/3\}. \quad (3.110)$$

Since we have three non-negative \mathcal{K} -exponents we see that this balance indeed corresponds to a general solution. The variables in this case expand as follows,

$$x = \alpha \Upsilon^{1/6} + 3/5 \alpha A \lambda c_{13}^2 \Upsilon^{5/6} - 27/14 \alpha A^2 \lambda^2 c_{13}^4 \Upsilon^{3/2} - 2 \alpha A c_{54} \Upsilon^{11/6} + \dots \quad (3.111)$$

$$y = \alpha/6 \Upsilon^{-5/6} + 1/2 \alpha A \lambda c_{13}^2 \Upsilon^{-1/6} - 81/28 \alpha A^2 \lambda^2 c_{13}^4 \Upsilon^{1/2} - 11/3 \alpha A c_{54} \Upsilon^{5/6} + \dots \quad (3.112)$$

$$z = c_{13} \Upsilon^{-2/3} - 12/5 A \lambda c_{13}^3 + 396/35 A^2 \lambda^2 c_{13}^5 \Upsilon^{2/3} + \dots \quad (3.113)$$

$$w = 1/(24A) \Upsilon^{-2} - 3/10 \lambda c_{13}^2 \Upsilon^{-4/3} + 747/350 A \lambda^2 c_{13}^4 \Upsilon^{-2/3} + c_{54} \Upsilon^{-1/3} + \dots \quad (3.114)$$

We ought to check the compatibility conditions for $j = 1$ and $j = 5$. We find

$$({}_{III}\mathcal{K}_2 - 1/3\mathcal{I})\mathbf{c}_1 = \begin{pmatrix} -1/2 & 1 & 0 & 0 \\ -5/36 & 1/2 & 0 & -10/3\alpha A \\ 0 & 0 & 0 & 0 \\ 1/(12\alpha A) & -1/(2\alpha A) & 0 & -1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_{13} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.115)$$

so that the compatibility condition for $j = 1$ is satisfied. Also for $j = 5$ we have

$$\begin{aligned} ({}_{III}\mathcal{K}_2 - 5/3\mathcal{I})\mathbf{c}_5 &= \begin{pmatrix} -11/6 & 1 & 0 & 0 \\ -5/36 & -5/6 & 0 & -10/3\alpha A \\ 0 & 0 & -4/3 & 0 \\ 1/(12\alpha A) & -1/(2\alpha A) & 0 & -5/3 \end{pmatrix} \begin{pmatrix} -2\alpha A c_{54} \\ -11/3 \alpha A c_{54} \\ c_{53} \\ c_{54} \end{pmatrix} = \\ &= \begin{pmatrix} 0 \\ 0 \\ -4/3 c_{53} \\ 0 \end{pmatrix} = P_5, \end{aligned} \quad (3.116)$$

where $c_{53} = 396/35 A^2 \lambda^2 c_{13}^5$. The corresponding eigenvector v is such that

$$v^\top = (1/(12A\alpha), -1/(2A\alpha), 0, 1).$$

The compatibility condition,

$$v^\top \cdot P_j = 0, \quad (3.117)$$

for $j = 5$ therefore holds true and Eqs. (3.111)-(3.114) represent the asymptotics around a collapse I singularity with the density of the fluid being divergent, i.e. as $\Upsilon \rightarrow 0$

$$a \rightarrow 0, \quad a' \rightarrow \infty, \quad \phi' \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (3.118)$$

3.5 Collapse type II singularity

We shall analyze in this Section the asymptotics represented by the balances $_{II}\mathcal{B}_3$ and $_{III}\mathcal{B}_4$ which suggest the emergence of a collapse type II singularity.

We start with the balance $_{II}\mathcal{B}_3$ that corresponds to a curved brane. The \mathcal{K} -matrix for this balance is given by

$$_{II}\mathcal{K}_3 = D\mathbf{f}^{(0)}(\alpha, \alpha, 0, 0) - \text{diag}(1, 0, r, -2), \quad (3.119)$$

where $D\mathbf{f}^{(0)}$ is the Jacobian matrix of the dominant part $\mathbf{f}^{(0)}$ in Eq. (3.21). The eigenvalues of $_{II}\mathcal{K}_3$ are

$$\text{spec}(_{II}\mathcal{K}_3) = \{-1, 0, -2 - 4\gamma, -4 - r\}. \quad (3.120)$$

For $\gamma = -1$ and $r = 1$, this balance becomes

$$_{II}\mathcal{B}_3 = \{(\alpha, \alpha, 0, 0), (1, 0, 1, -2)\}, \quad (3.121)$$

and the eigenvalues of the \mathcal{K} -matrix are

$$\text{spec}(_{II}\mathcal{K}_3) = \{-1, 0, 2, -5\}. \quad (3.122)$$

We may set the arbitrary constant appearing for $j = -5$ equal to zero and find a particular solution around a collapse II singularity. The asymptotic expansions of the variables now are:

$$x = \alpha\Upsilon + A\alpha/9c_{24}\Upsilon^3 + \dots \quad (3.123)$$

$$y = \alpha + A\alpha/3c_{24}\Upsilon^2 + \dots \quad (3.124)$$

$$z = 0 + \dots \quad (3.125)$$

$$w = c_{24} + \dots \quad (3.126)$$

The compatibility condition for $j = 2$ is satisfied since

$$({}_{II}\mathcal{K}_3 - 2\mathcal{I})\mathbf{c}_2 = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & (2/3)\alpha A \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\alpha A/9)c_{24} \\ (\alpha A/3)c_{24} \\ 0 \\ c_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.127)$$

It follows then from Eqs. (3.123)-(3.126) that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow c_{24}. \quad (3.128)$$

Next we turn to the balance ${}_{III}\mathcal{B}_4$. This balance corresponds to a potentially general solution of a curved brane. The eigenvalues of its \mathcal{K} -matrix are

$$\text{spec}({}_{III}\mathcal{K}_4) = \{-3, -2 - 4\gamma, -1, 0\}. \quad (3.129)$$

For $\gamma < -1/2$ we have one positive eigenvalue (the case $\gamma > -1/2$ for which we have three negative eigenvalues is considered later in Subsection 3.7). We may set $\mathbf{c}_{-3} = 0$ from the beginning and find an asymptotic expansion of a particular solution around a collapse II singularity. We choose $\gamma = -3/4$. Then

$$\text{spec}({}_{III}\mathcal{K}_4) = \{-3, 1, -1, 0\}, \quad (3.130)$$

and we find the following asymptotic forms

$$x = \alpha\Upsilon + A\alpha/6c_{14}\Upsilon^2 + \dots \quad (3.131)$$

$$y = \alpha + A\alpha/3c_{14}\Upsilon + \dots \quad (3.132)$$

$$z = 0 + \dots \quad (3.133)$$

$$w = c_{14}\Upsilon^{-1} + \dots \quad (3.134)$$

The compatibility condition for $j = 1$ is trivially satisfied since

$$({}_{III}\mathcal{K}_4 - \mathcal{I})\mathbf{c}_1 = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & \alpha A/3 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\alpha A/6)c_{14} \\ (\alpha A/3)c_{14} \\ 0 \\ c_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.135)$$

The forms in Eqs. (3.131)-(3.134) then imply that as $\Upsilon \rightarrow 0$,

$$a \rightarrow 0, \quad a' \rightarrow \alpha, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow \infty. \quad (3.136)$$

Thus for a curved brane we found collapse II singularities with $\phi' \rightarrow 0$ and $\rho_2 \rightarrow \infty, \rho_s$, $\rho_s \neq 0$. This means that asymptotically the leak of energy from the brane and into the bulk is controlled solely by the fluid.

3.6 Big rip singularity

In this Section we examine the balances $_{II}\mathcal{B}_1$ and $_{III}\mathcal{B}_2$ which correspond to a flat brane and describe the asymptotics around a big rip singularity when we choose a value of γ such that $\gamma < -1$.

The first of these balances has the following \mathcal{K} -exponents

$$\text{spec}(_{II}\mathcal{K}_1) = \{-1, 0, 0, -2(p-1)\}. \quad (3.137)$$

Choosing $\gamma = -2$, we have that

$$_{II}\mathcal{B}_1 = \{(\alpha, -\alpha/2, c, 3/(8A)), (-1/2, -3/2, 2, -2)\} \quad (3.138)$$

and

$$_{II}\mathcal{K}_1 = \{-1, 0, 0, 3\}. \quad (3.139)$$

This leads to the following solution,

$$x = \alpha\Upsilon^{-1/2} - \alpha/(4c)c_{33}\Upsilon^{5/2} + \dots \quad (3.140)$$

$$y = -\alpha/2\Upsilon^{-3/2} - 5\alpha/(8c)c_{33}\Upsilon^{3/2} + \dots \quad (3.141)$$

$$z = c\Upsilon^2 + c_{33}\Upsilon^5 + \dots \quad (3.142)$$

$$w = 3/(8A)\Upsilon^{-2} - 3/(8Ac)c_{33}\Upsilon + \dots \quad (3.143)$$

We validate the compatibility condition for $j = 3$, since

$$({}_{II}\mathcal{K}_1 - 3\mathcal{I})\mathbf{c}_3 = \begin{pmatrix} -5/2 & 1 & 0 & 0 \\ 3/4 & -3/2 & 0 & 2\alpha A \\ -2c/\alpha & -4c/\alpha & -3 & 0 \\ 3/(4\alpha A) & 3/(2\alpha A) & 0 & -3 \end{pmatrix} \begin{pmatrix} -\alpha/(4c)c_{33} \\ -5\alpha/(8c)c_{33} \\ c_{33} \\ -3/(8Ac)c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.144)$$

As $\Upsilon \rightarrow 0$ we have a big rip singularity:

$$a \rightarrow \infty, \quad a' \rightarrow -\infty, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow \infty. \quad (3.145)$$

We now examine the second balance ${}_{III}\mathcal{B}_2$. The \mathcal{K} -exponents for this balance are given by Eq. (3.108). We are interested in the case $\gamma < -1$ which we have not examined yet and for which we get two positive \mathcal{K} -exponents. As we show below this case leads to the emergence of a big rip singularity. We set for concreteness $\gamma = -3/2$. Then

$${}_{III}\mathcal{B}_2 = \{(\alpha, -\alpha, 0, 3/(2A)), (-1, -2, -1, -2)\} \quad (3.146)$$

and

$${}_{III}\mathcal{K}_2 = \{-1, 0, 4, 5\}. \quad (3.147)$$

In this case we have three non-negative \mathcal{K} -exponents so that this balance indeed corresponds to a general solution. The variables in this case expand as follows,

$$x = \alpha\Upsilon^{-1} + A\alpha/3c_{44}\Upsilon^3 + \dots \quad (3.148)$$

$$y = -\alpha\Upsilon^{-2} + A\alpha c_{44}\Upsilon^2 + \dots \quad (3.149)$$

$$z = c_{53}\Upsilon^4 + \dots \quad (3.150)$$

$$w = 3/(2A)\Upsilon^{-2} + c_{44}\Upsilon^2 + \dots \quad (3.151)$$

We ought to check the compatibility conditions for $j = 4$ and $j = 5$. We find

$$({}_{III}\mathcal{K}_2 - 4\mathcal{I})\mathbf{c}_4 = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 2 & -2 & 0 & (4/3)\alpha A \\ 0 & 0 & -1 & 0 \\ 3/(\alpha A) & 3/(\alpha A) & 0 & -4 \end{pmatrix} \begin{pmatrix} A\alpha/3c_{44} \\ A\alpha c_{44} \\ 0 \\ c_{44} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.152)$$

and

$$({}_{III}\mathcal{K}_2 - 5\mathcal{I})\mathbf{c}_5 = \begin{pmatrix} -4 & 1 & 0 & 0 \\ 2 & -3 & 0 & (4/3)\alpha A \\ 0 & 0 & 0 & 0 \\ 3/(\alpha A) & 3/(\alpha A) & 0 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_{53} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.153)$$

so that Eqs. (3.148)-(3.151) represent the asymptotics around a big rip singularity i.e. as $\Upsilon \rightarrow 0$

$$a \rightarrow \infty, \quad a' \rightarrow -\infty, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow \infty. \quad (3.154)$$

Hence in both of the cases studied in this Section we found that for a flat brane, a big rip singularity develops at a finite distance. We note here that the exchange of energy from the brane into the bulk is totally monitored by the fluid.

3.7 Behavior at infinity

In this Section we consider the balances ${}_I\mathcal{B}_3$, ${}_{III}\mathcal{B}_4$, ${}_I\mathcal{B}_4$, ${}_{II}\mathcal{B}_2$ and ${}_{III}\mathcal{B}_2$ that offer the possibility of escaping the finite-distance singularities met before and describe the behavior of our model at infinity.

We begin with the analysis of the balance ${}_I\mathcal{B}_3$. The eigenvalues of its \mathcal{K} -matrix read

$$\text{spec}({}_I\mathcal{K}_3) = \{-1, -3, 0, 0\}, \quad (3.155)$$

hence we may expand (x, y, z, w) in descending powers in order to meet the arbitrary constants appearing at $j = -1$ and $j = -3$. We choose $\gamma = -1/2$. The balance ${}_I\mathcal{B}_3$ then

corresponds to a general solution of a flat or curved brane. In particular we find

$$x = \alpha \Upsilon + c_{-11} + \dots \quad (3.156)$$

$$y = \alpha + \dots \quad (3.157)$$

$$z = c_{-33} \Upsilon^{-4} + \dots \quad (3.158)$$

$$w = \zeta \Upsilon^{-2} - 2\zeta/\alpha c_{-11} \Upsilon^{-3} + 3\zeta/\alpha^2 c_{-11}^2 \Upsilon^{-4} - 4\zeta/\alpha^3 c_{-11}^3 \Upsilon^{-5} + \dots \quad (3.159)$$

The compatibility conditions for $j = -1$ are satisfied since

$$({}_I\mathcal{K}_3 + \mathcal{I})\mathbf{c}_{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 2\zeta/\alpha & -2\zeta/\alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{-11} \\ 0 \\ 0 \\ -2\zeta/\alpha c_{-11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.160)$$

For $j = -3$ we find

$$({}_I\mathcal{K}_3 + 3\mathcal{I})\mathbf{c}_{-3} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\zeta/\alpha & -2\zeta/\alpha & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_{-33} \\ -4\zeta/\alpha^3 c_{-11}^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -12\zeta/\alpha^3 c_{-11}^3 \end{pmatrix} = P_{-3}. \quad (3.161)$$

Since a corresponding eigenvector here is $v^\top = (0, 0, 1, 0)$ and

$$v^\top \cdot P_{-3} = 0 \quad (3.162)$$

the compatibility condition for $j = -3$ is also satisfied and Eqs. (3.156)-(3.159) then imply that as $\Upsilon \rightarrow \infty$

$$a \rightarrow \infty, \quad a' \rightarrow \alpha, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow 0. \quad (3.163)$$

We now examine the balance ${}_{III}\mathcal{B}_4 = \{(\alpha, \alpha, 0, 0), (1, 0, -1, -2)\}$ that corresponds to a general solution of a curved brane. The \mathcal{K} -exponents are given by Eq. (3.129). For $\gamma > -1/2$ we have three negative \mathcal{K} -exponents. We choose $\gamma = 0$. Then

$$\text{spec}({}_{III}\mathcal{K}_4) = \{-3, -2, -1, 0\}. \quad (3.164)$$

We find here that

$$x = \alpha \Upsilon + c_{-11} - (1/3)\alpha A c_{-24} \Upsilon^{-1} - (5/9)A c_{-11} c_{-24} \Upsilon^{-2} + \dots \quad (3.165)$$

$$y = \alpha + (1/3)\alpha A c_{-24} \Upsilon^{-2} + (10/9)A c_{-11} c_{-24} \Upsilon^{-3} + \dots \quad (3.166)$$

$$z = c_{-33} \Upsilon^{-4} + \dots \quad (3.167)$$

$$w = c_{-24} \Upsilon^{-4} + (4/\alpha) c_{-11} c_{-24} \Upsilon^{-5} + \dots \quad (3.168)$$

We check the compatibility conditions for $j = -1$, $j = -2$ and $j = -3$. For $j = -1$ we find

$$({}_{III}\mathcal{K}_4 + \mathcal{I})\mathbf{c}_{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -(2/3)\alpha A \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_{-11} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.169)$$

while for $j = -2$

$$({}_{III}\mathcal{K}_4 + 2\mathcal{I})\mathbf{c}_{-2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -(2/3)\alpha A \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -(1/3)\alpha A c_{-24} \\ (1/3)\alpha A c_{-24} \\ 0 \\ c_{-24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.170)$$

For $j = -3$ we have that

$$\begin{aligned} ({}_{III}\mathcal{K}_4 + 3\mathcal{I})\mathbf{c}_{-3} &= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & -(2/3)\alpha A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -(5/9)A c_{-11} c_{-24} \\ (10/9)A c_{-11} c_{-24} \\ c_{-33} \\ (4/\alpha) c_{-11} c_{-24} \end{pmatrix} = \\ &= \begin{pmatrix} 0 \\ 2/3 A c_{-11} c_{-24} \\ 0 \\ 4/\alpha c_{-11} c_{-24} \end{pmatrix} = P_{-3}, \end{aligned} \quad (3.171)$$

and an eigenvector v is such that $v^\top = (0, 0, 1, 0)$. The compatibility condition,

$$v^\top \cdot P_{-3} = 0, \quad (3.172)$$

therefore holds true. Eqs. (3.165)-(3.168) then imply that as $\Upsilon \rightarrow \infty$

$$a \rightarrow \infty, \quad a' \rightarrow \alpha, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow 0. \quad (3.173)$$

On the other hand, the \mathcal{K} -exponents for the balance ${}_I\mathcal{B}_4$, that also corresponds to a curved brane, are

$$\text{spec}({}_I\mathcal{K}_4) = \{-3, -s - 4(\gamma + 1), -1, 0\}. \quad (3.174)$$

For $s = -3$ and $\gamma = 1/4$ we find the following behavior

$$x = \alpha\Upsilon + c_{-11} - (1/6)\alpha A c_{-24}\Upsilon^{-2} + \dots \quad (3.175)$$

$$y = \alpha + (1/3)\alpha A c_{-24}\Upsilon^{-3} + \dots \quad (3.176)$$

$$z = c_{-33}\Upsilon^{-4} + \dots \quad (3.177)$$

$$w = c_{-24}\Upsilon^{-5} - (5/\alpha)c_{-11}c_{-24}\Upsilon^{-6} + \dots \quad (3.178)$$

The compatibility conditions for $j = -1$ and $j = -2$ are trivially satisfied since

$$({}_I\mathcal{K}_4 + \mathcal{I})\mathbf{c}_{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_{-11} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.179)$$

and

$$({}_I\mathcal{K}_4 + 2\mathcal{I})\mathbf{c}_{-2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_{-24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.180)$$

For $j = -3$ we have that

$$\begin{aligned}
({}_I\mathcal{K}_4 + 3\mathcal{I})\mathbf{c}_{-3} &= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -(1/6)A\alpha c_{-24} \\ (1/3)A\alpha c_{-24} \\ c_{-33} \\ -(5/\alpha)c_{-11}c_{-24} \end{pmatrix} = \\
&= \begin{pmatrix} 0 \\ A\alpha c_{-24} \\ 0 \\ -(5/\alpha)c_{-11}c_{-24} \end{pmatrix} = P_{-3}, \tag{3.181}
\end{aligned}$$

and an eigenvector v is such that $v^\top = (0, 0, 1, 0)$ so that the compatibility condition for $j = -3$,

$$v^\top \cdot P_{-3} = 0, \tag{3.182}$$

is also satisfied. Therefore Eqs. (3.175)-(3.178) then imply that as $\Upsilon \rightarrow \infty$

$$a \rightarrow \infty, \quad a' \rightarrow \alpha, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow 0. \tag{3.183}$$

Let us now examine the balance ${}_{II}\mathcal{B}_2$ that corresponds to a flat brane. For $\gamma = -3/4$ (hence $p = 2$) and $r = -5$ this balance reads ${}_{II}\mathcal{B}_2 = \{(\alpha, 2\alpha, 0, 6/A), (2, 1, -5, -2)\}$. and the eigenvalues of the ${}_{II}\mathcal{K}_2$ matrix are

$$\text{spec}({}_{II}\mathcal{K}_2) = \{-3, -2, -1, 0\}. \tag{3.184}$$

We find here the following asymptotic behavior

$$x = \alpha\Upsilon^2 - A\alpha/6c_{-14}\Upsilon - A\alpha/6c_{-24} + A^2\alpha/36c_{-14}^2 + \dots \tag{3.185}$$

$$y = 2\alpha\Upsilon - A\alpha/6c_{-14} + \dots \tag{3.186}$$

$$z = c_{-33}\Upsilon^{-8} + \dots \tag{3.187}$$

$$w = 6/A\Upsilon^{-2} + c_{-14}\Upsilon^{-3} + c_{-24}\Upsilon^{-4} + (-A^2/36c_{-14}^3 + A/3c_{-14}c_{-24})\Upsilon^{-5} + \dots \tag{3.188}$$

The compatibility conditions for $j = -1$, $j = -2$ and $j = -3$ are satisfied. Particularly for $j = -1$ we find

$$({}_{II}\mathcal{K}_2 + \mathcal{I})\mathbf{c}_{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & A\alpha/3 \\ 0 & 0 & -2 & 0 \\ 12/(A\alpha) & -6/(A\alpha) & 0 & 1 \end{pmatrix} \begin{pmatrix} -A\alpha/6c_{-14} \\ -A\alpha/6c_{-14} \\ 0 \\ c_{-14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.189)$$

For $j = -2$ we find that

$$\begin{aligned} ({}_{II}\mathcal{K}_2 + 2\mathcal{I})\mathbf{c}_{-2} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & A\alpha/3 \\ 0 & 0 & -1 & 0 \\ 12/(A\alpha) & -6/(A\alpha) & 0 & 2 \end{pmatrix} \begin{pmatrix} -A\alpha/6c_{-24} + A^2\alpha/36c_{-14}^2 \\ 0 \\ 0 \\ c_{-24} \end{pmatrix} = \\ &= \begin{pmatrix} 0 \\ A^2\alpha/18c_{-14}^2 \\ 0 \\ A/3c_{-14}^2 \end{pmatrix} = P_{-2}, \end{aligned} \quad (3.190)$$

and an eigenvector v is such that

$$v^\top = (12/(A\alpha), -6/(A\alpha), 0, 1) \quad (3.191)$$

and hence we find

$$v^\top \cdot P_{-2} = 0, \quad (3.192)$$

which means that the compatibility condition is satisfied also for $j = -2$. Finally, for $j = -3$ we see that

$$({}_{II}\mathcal{K}_2 + 3\mathcal{I})\mathbf{c}_{-3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & A\alpha/3 \\ 0 & 0 & 0 & 0 \\ 12/(A\alpha) & -6/(A\alpha) & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_{-33} \\ -A^2/36c_{-14}^3 + A/3c_{-14}c_{-24} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ A\alpha/3(-A^2/36c_{-14}^3 + A/3c_{-14}c_{-24}) \\ 0 \\ 3(-A^2/36c_{-14}^3 + A/3c_{-14}c_{-24}) \end{pmatrix} = P_{-3} \quad (3.193)$$

while an eigenvector v is such that

$$v^\top = (0, 0, 1, 0), \quad (3.194)$$

and hence

$$v^\top \cdot P_{-3} = 0, \quad (3.195)$$

so that the compatibility condition for $j = -3$ is satisfied. We thus see from Eqs. (3.185)-(3.188) that as $\Upsilon \rightarrow \infty$

$$a \rightarrow \infty, \quad a' \rightarrow \infty, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow 0. \quad (3.196)$$

We now move on to the balance $_{III}\mathcal{B}_2$ which also corresponds to a flat brane. It follows from (3.108) that for $-1 < \gamma < -1/2$ this balance leads to three negative \mathcal{K} -exponents. We take $\gamma = -3/4$ and then this balance becomes

$$_{III}\mathcal{B}_2 = \{(\alpha, 2\alpha, 0, 6/A), (2, 1, -1, -2)\}, \quad (3.197)$$

and the eigenvalues of the $_{III}\mathcal{K}_2$ matrix are

$$\text{spec}(_{III}\mathcal{K}_2) = \{-7, -2, -1, 0\}. \quad (3.198)$$

In this case we find that

$$x = \alpha\Upsilon^2 + c_{-11}\Upsilon + c_{-21} + \dots \quad (3.199)$$

$$y = 2\alpha\Upsilon + c_{-11} + \dots \quad (3.200)$$

$$z = c_{-73}\Upsilon^{-8} + \dots \quad (3.201)$$

$$\begin{aligned} w &= 6/A\Upsilon^{-2} - 6/(A\alpha)c_{-11}\Upsilon^{-3} + 6/(A\alpha^2)(c_{-11}^2 - \alpha c_{-21})\Upsilon^{-4} + \\ &+ c_{-34}\Upsilon^{-5} + c_{-44}\Upsilon^{-6} + c_{-54}\Upsilon^{-7} + c_{-64}\Upsilon^{-8} + c_{-74}\Upsilon^{-9} + \dots, \end{aligned} \quad (3.202)$$

where c_{j4} with $j = -3, \dots, -7$, are polynomials in α , c_{-11} and c_{-21} . The compatibility condition at $j = -1$ is trivially satisfied since we have that

$$({}_{III}\mathcal{K}_2 + \mathcal{I})\mathbf{c}_{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & A\alpha/3 \\ 0 & 0 & -6 & 0 \\ 12/(A\alpha) & -6/(A\alpha) & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{-11} \\ c_{-11} \\ 0 \\ -6/(A\alpha)c_{-11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.203)$$

For $j = -2$ we find that

$$({}_{III}\mathcal{K}_2 + 2\mathcal{I})\mathbf{c}_{-2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & A\alpha/3 \\ 0 & 0 & -5 & 0 \\ 12/(A\alpha) & -6/(A\alpha) & 0 & 2 \end{pmatrix} \begin{pmatrix} c_{-21} \\ 0 \\ 0 \\ 6/(A\alpha^2)(c_{-11}^2 - \alpha c_{-21}) \end{pmatrix} = \quad (3.204)$$

$$= \begin{pmatrix} 0 \\ 2/\alpha c_{-11}^2 \\ 0 \\ 12/(A\alpha^2)c_{-11}^2 \end{pmatrix} = P_{-2}, \quad (3.205)$$

and $v^\top = (12/(A\alpha), -6/(A\alpha), 0, 1)$ so that the compatibility condition

$$v^\top \cdot P_{-2} = 0$$

is satisfied. Finally, for $j = -7$ we have that

$$({}_{III}\mathcal{K}_2 + 7\mathcal{I})\mathbf{c}_{-7} = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 2 & 6 & 0 & A\alpha/3 \\ 0 & 0 & 0 & 0 \\ 12/(A\alpha) & -6/(A\alpha) & 0 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_{-73} \\ c_{-74} \end{pmatrix} = \begin{pmatrix} 0 \\ A\alpha/3c_{-74} \\ 0 \\ 7c_{-74} \end{pmatrix} = P_{-7}, \quad (3.206)$$

and the corresponding eigenvector here is such that $v^\top = (0, 0, 1, 0)$ which implies that the compatibility condition

$$v^\top \cdot P_{-7} = 0$$

holds true. From Eqs. (3.199)-(3.202) it follows that as $\Upsilon \rightarrow \infty$

$$a \rightarrow \infty, \quad a' \rightarrow \infty, \quad \phi' \rightarrow 0, \quad \rho_2 \rightarrow 0. \quad (3.207)$$

In our previous work in [2], the bulk was filled entirely by the fluid and we found that for $-1 < \gamma < -1/2$ and for a flat brane the avoidance of finite-distance singularities was the *only* possible asymptotic behavior. This fact suggested that a self-tuning mechanism could be build within the framework of such model, a property that we would anticipate to hold also in the more complicated case studied in this paper. However, by the analysis we did so far, we see instead that although the balance $_{III}\mathcal{B}_2$ implies a behavior that is singular only at infinite distance, the balance $_{III}\mathcal{B}_3$ that is also valid for this range of γ implies a singular (collapse type I) behavior at finite distance (as this follows from Eqs. (3.85)-(3.88) in Subsection 3.4.2.). Both of these balances represent behaviors of the general solution. In particular, the balance $_{III}\mathcal{B}_2$ describes the asymptotic behavior of the general solution in a neighborhood of infinity, while, the balance $_{III}\mathcal{B}_3$ describes the asymptotic behavior of the general solution around a finite-distance singularity. We thus conclude that in this case the avoidance of finite-distance singularities for flat brane becomes impossible. Our results suggest that the singular behavior encountered here is driven by the presence of the scalar field which is left to act independently from the fluid. What would happen, instead, if it interacted with the fluid? In the next Section, we show that by choosing the interaction parameters in an adequate way, we may resolve this unwanted situation and recover the possibility of avoiding finite-distance singularities.

4 Interacting mixture in the bulk

In this Section, we study the possible behaviors that arise when the two bulk components interact with each other. We begin by searching to find what are the forms of the balances that are possible in this more intricate case. In order to simplify our calculations, that are much more complicated than in the case studied previously, we may set to zero either one of the two parameters σ and ν that define the interaction and let the remaining one

vary arbitrarily. If we choose both parameters nonzero we are led to balances $\{\mathbf{a}, \mathbf{p}\}$ with the exponents in the vector \mathbf{p} being irrational which leads to the existence of logarithms in the series expansions of the variables. In the next paragraphs we show that the choice that gives the desired result, meaning the avoidance of singularities, is $\sigma = 0$.⁴

We start the analysis by putting $\sigma = 0$ in the system (2.11)-(2.14) and letting ν be arbitrary but nonzero. We consider all terms dominant and by substituting Eq. (3.10) there, we find the following four balances

$${}_{\nu}\mathcal{B}_1 = \left\{ \left(\alpha, \frac{2\alpha}{8+\nu}, \sqrt{\frac{3(4\gamma-4-\nu)}{A\lambda(\gamma-1)(8+\nu)^2}}, \frac{3\nu}{2A(\gamma-1)(8+\nu)^2} \right), \left(\frac{2}{8+\nu}, -\frac{6+\nu}{8+\nu}, -1, -2 \right) \right\} \quad (4.1)$$

$${}_{\nu}\mathcal{B}_2 = \left\{ \left(\alpha, \alpha p, 0, \frac{3p^2}{2A} \right), (p, p-1, -1, -2) \right\}, \quad p = \frac{1}{2(\gamma+1)}, \quad \gamma \neq -1, -1/2, \quad (4.2)$$

$${}_{\nu}\mathcal{B}_3 = \{(\alpha, \alpha, 0, 0), (1, 0, -1, -2)\}, \quad (4.3)$$

$${}_{\nu}\mathcal{B}_4 = \{(\alpha, 0, 0, 0), (0, -1, -1, -2)\}. \quad (4.4)$$

The balance ${}_{\nu}\mathcal{B}_1$ is valid for $\nu \neq -6, -8, 4\gamma - 4$ and $\gamma \neq 1$ and because of the square root we may have either $\nu > 4\gamma - 4$ and $\gamma < 1$, or, $\nu < 4\gamma - 4$ and $\gamma > 1$. Substitution of these balances in the constraint equation (2.15) shows that ${}_{\nu}\mathcal{B}_1$, ${}_{\nu}\mathcal{B}_2$ and ${}_{\nu}\mathcal{B}_4$ correspond to a flat brane whereas ${}_{\nu}\mathcal{B}_3$ corresponds to a curved brane with α satisfying $\alpha^2 = kH^2$.

⁴The case $\nu = 0$ and σ arbitrary does not lead to avoidance of singularities but instead it brings back the same problem we faced in Subsection 3.7.

We now calculate the Jacobian of the vector field (2.16) (with $\sigma = 0$) and find

$$\begin{aligned}
D\mathbf{f}(x, y, z, w) = & \\
= & \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{2}{3}A(1+2\gamma)w - \lambda Az^2 & 0 & -2\lambda Azx & -\frac{2}{3}A(1+2\gamma)x \\ \left(4 + \frac{\nu}{2}\right) \frac{yz}{x^2} & -\left(4 + \frac{\nu}{2}\right) \frac{z}{x} & -\left(4 + \frac{\nu}{2}\right) \frac{y}{x} & 0 \\ 4(\gamma+1) \frac{yw}{x^2} - \frac{\nu\lambda}{2} \frac{yz^2}{x^2} & -4(\gamma+1) \frac{w}{x} + \frac{\nu\lambda}{2} \frac{z^2}{x} & \nu\lambda \frac{yz}{x} & -4(\gamma+1) \frac{y}{x} \end{pmatrix}.
\end{aligned} \tag{4.5}$$

The balance ${}_{\nu}\mathcal{B}_4$ is discarded because it does not give the -1 \mathcal{K} -exponent, but it has instead

$$\text{spec}({}_{\nu}\mathcal{K}_4) = \{0, 1, 1, 2\}. \tag{4.6}$$

The balance ${}_{\nu}\mathcal{B}_3$, on the other hand, has

$$\text{spec}({}_{\nu}\mathcal{K}_3) = \{-1, 0, -2(1+2\gamma), -3 - \nu/2\}, \tag{4.7}$$

which implies that for a curved brane we may avoid the finite-distance singularities for $\gamma > -1/2$ and $\nu > -6$. We may also include in this range of γ the value $-1/2$ since then the following balance arises:

$${}_{\nu}\mathcal{B}_{-1/2} = \{(\alpha, \alpha, 0, \delta), (1, 0, -1-2)\}. \tag{4.8}$$

The balance ${}_{\nu}\mathcal{B}_{-1/2}$ corresponds to a general solution of a flat or curved brane with $\delta = 3/(2A)(1 - kH^2/\alpha^2)$, $\delta \neq 0$ and gives

$$\text{spec}({}_{\nu}\mathcal{K}_{-1/2}) = \{-1, 0, 0, -3 - \nu/2\}, \tag{4.9}$$

so that for $\nu > -6$ finite-distance singularities can be avoided.

The \mathcal{K} -exponents for the balance ${}_{\nu}\mathcal{B}_1$ are

$$\text{spec}({}_{\nu}\mathcal{K}_1) = \left\{ -1, 0, \frac{2(6 + \nu)}{8 + \nu}, \frac{2(4 - 4\gamma + \nu)}{8 + \nu} \right\}, \quad (4.10)$$

while for the balance ${}_{\nu}\mathcal{B}_2$ we find,

$$\text{spec}({}_{\nu}\mathcal{K}_2) = \left\{ -1, 0, \frac{1 + 2\gamma}{1 + \gamma}, \frac{-4 + 4\gamma - \nu}{4 + 4\gamma} \right\}. \quad (4.11)$$

The balance ${}_{\nu}\mathcal{B}_2$ implies that we may escape finite-distance singularities for $-1 < \gamma < -1/2$ and $\nu > -4 + 4\gamma$ since then the last two eigenvalues of ${}_{\nu}\mathcal{K}_2$ become negative. We note though that for these ranges of γ and ν the balance ${}_{\nu}\mathcal{B}_1$ is also valid and it is such that at least the last eigenvalue of ${}_{\nu}\mathcal{K}_1$ becomes then positive. We may exploit this fact by keeping γ in the interval $(-1, -1/2)$ and restricting ν to fall in $(-4 + 4\gamma, -6)$. For this choice of parameters, the last two eigenvalues of ${}_{\nu}\mathcal{K}_1$ have opposite signs which means that the solution described by ${}_{\nu}\mathcal{B}_1$ is neither valid around a finite-distance singularity, nor, around a neighborhood of infinity but rather in the limited area of an annulus failing thus to provide us with any substantial information about the asymptotics of the dynamical system (2.11)-(2.14). We are then left with the unique possibility described by the balance ${}_{\nu}\mathcal{B}_2$ which is the asymptotic expansion of the general solution of the system in a neighborhood of infinity. Consequently, avoidance of finite-distance singularities is feasible for $-1 < \gamma < -1/2$ and $-4 + 4\gamma < \nu < -6$.

5 Conclusions

We have studied a model consisting of a three-brane embedded in a five-dimensional bulk filled with a scalar field and an analog to a perfect fluid possessing a general equation of state $P_2 = \gamma\rho_2$, characterized by the constant parameter γ . The two bulk matter components may act independently, or, they may interact with each other by exchanging energy in a way that the total energy is conserved.

We have started off by analyzing the evolution of our model in the case that the two bulk matter components do not interact with each other (the two interaction parameters

are set to zero in this case). We have shown that a quite general feature of the asymptotic behavior of such model is the emergence of a finite-distance singularity that is of the collapse type I, II or big rip class. The singularities accommodated in the first two classes share common characteristics such as the vanishing of the warp factor. However, the derivative of the warp factor behaves differently in each case: it is divergent in the collapse type I class whereas it remains finite in the collapse type II class. The collapse type I singularity may arise for all values of γ , whereas, the collapse type II class arises only when $\gamma < -1/2$ as this is illustrated by the balance $_{III}\mathcal{B}_4$ but also by the balances $_{II}\mathcal{B}_3$, $_{I}\mathcal{B}_3$ and $_{I}\mathcal{B}_4$ for an adequate choice of their parameters. All of these balances lead to particular solutions for this range of γ . The third class, on the other hand, of big rip singularities, arises always with $\gamma < -1$ and it is characterized by the divergence of the warp factor, its derivative and density of the fluid while the energy density of the scalar field now tends to zero in the neighborhood of the singularity.

We completed our analysis for non-interacting bulk matter by addressing the important issue of whether it is possible to avoid finite-distance singularities. We found that this is true only for a curved brane and for $\gamma > -1/2$. This is demonstrated by the balances $_{III}\mathcal{B}_4$, $_{II}\mathcal{B}_3$ and $_{I}\mathcal{B}_4$ that give negative \mathcal{K} -exponents for $\gamma > -1/2$. For a flat brane on the other hand, the avoidance of singularities is not possible, as this was discussed in detail in the last paragraph of Subsection 3.7. The main reason for the failure to escape finite-distance singularities in the case of flat brane is that for all values of γ there always exists a balance that corresponds to a general solution and describes its behavior around finite-distance singularities due to the presence of the scalar field component in the bulk. It is thus impossible to find ranges of γ that they are not characterized by singular behavior.

For illustration, we present a summary of our results for this first analysis in the Table 1 below, using the notation for the various singularities introduced in Section 2 and the symbol $*$ to denote a balance that corresponds to a particular solution. For each entry in the table we have taken into account all the corresponding examples in our

analysis to deduce the form of each balance and \mathcal{K} -exponents. Note that we used the numerical examples only as representatives of the corresponding asymptotic behaviors for the different regions of the parameter γ . We also give the ranges of r and s , entering in the definitions of the balances around the type I and II singularities defined in Subsections 3.1 and 3.2, that lead to the most general behavior possible for each balance.

We continued our analysis to include also cases with interacting bulk matter, motivated by the fact that it is impossible to find in our flat brane model ranges of γ that lead to avoidance of singularities within finite distance in the case of non-interacting bulk matter. We studied the case of interaction $\sigma = 0$, ν arbitrary. We have shown that this choice leads to the avoidance of finite-distance singularities for $-1 < \gamma < -1/2$ and $-4 + 4\gamma < \nu < -6$. For a curved brane, avoidance of finite-distance singularities is allowed for $\gamma \geq -1/2$, $\nu > -6$ and $\sigma = 0$. These results enforce our previous conclusion about the possibility of such solutions and show that they are also possible in the more involved case considered in this paper, thus proving that the self-tuning mechanism may be sustained under field interactions in the bulk.

We illustrate the results we found for the case of non-interacting bulk components as well as for the case with interaction $\sigma = 0$ and ν arbitrary in the following tables:

equation of state	flat brane		curved brane	
$P_2 = \gamma \rho_2$	type	balance	type	balance
$\gamma > 1$	singular type I	$_{II}\mathcal{B}_1, _{III}\mathcal{B}_2,$ $_{II}\mathcal{B}_2 \ (-1 < r < -2/(\gamma + 1))$	<i>regular</i>	at ∞ : $_{III}\mathcal{B}_4,$ $_I\mathcal{B}_4 \ (-4(\gamma + 1) < s < -2),$ $_{II}\mathcal{B}_3 \ (-4 < r < -1)$
$\gamma = 1$	singular type I	$_{III}\mathcal{B}_1$	<i>regular</i>	at ∞ : $_{III}\mathcal{B}_4,$ $_I\mathcal{B}_4 \ (-8 < s < -2),$ $_{II}\mathcal{B}_3 \ (-4 < r < -1)$
$-1/2 < \gamma < 1$	singular type I	$_I\mathcal{B}_1, _{III}\mathcal{B}_3,$ $_I\mathcal{B}_2 \ (-2 < s < -(1 + \gamma))$	<i>regular</i>	at ∞ : $_{III}\mathcal{B}_4,$ $_I\mathcal{B}_4 \ (-4(\gamma + 1) < s < -2),$ $_{II}\mathcal{B}_3 \ (-4 < r < -1)$
$\gamma = -1/2$	<i>regular</i> singular type I	at ∞ $_I\mathcal{B}_3$ $_I\mathcal{B}_2 \ (s < -1/2), _{III}\mathcal{B}_3$	<i>regular</i> singular type I	at ∞ : $_I\mathcal{B}_3, _I\mathcal{B}_4 \ (s > -2)$ $_I\mathcal{B}_1$
$-1 < \gamma < -1/2$	<i>regular</i> singular type I	at ∞ : $_{III}\mathcal{B}_2,$ $_{II}\mathcal{B}_2 \ (-2/(\gamma + 1) < r < -1)$ $_I\mathcal{B}_1, _{III}\mathcal{B}_3,$ $_I\mathcal{B}_2 \ (-2 < s < -(1 + \gamma))$	<i>regular</i> singular type II	$_{III}\mathcal{B}_4^*, _I\mathcal{B}_4^* \ (s < -2),$ $_{II}\mathcal{B}_3^* \ (-4 < r < -1)$ $_I\mathcal{B}_4^* \ (-2 < s < -4(1 + \gamma)),$ $_{III}\mathcal{B}_4^*, _I\mathcal{B}_3^*, _{II}\mathcal{B}_3^* \ (r > -1)$
$\gamma = -1$	singular type I	$_I\mathcal{B}_1, _{III}\mathcal{B}_3,$ $_I\mathcal{B}_2 \ (-2 < s < 0)$	<i>regular</i> singular type II	at ∞ : $_I\mathcal{B}_4^* \ (s < -2), _{III}\mathcal{B}_4^*,$ $_{II}\mathcal{B}_3^* \ (-4 < r < -1)$ $_I\mathcal{B}_4^* \ (-2 < s < -4(1 + \gamma)),$ $_{III}\mathcal{B}_4^*, _I\mathcal{B}_3^*, _{II}\mathcal{B}_3^* \ (r > -1)$
$\gamma < -1$	singular big rip singular type I	$_{II}\mathcal{B}_1, _{III}\mathcal{B}_2,$ $_{II}\mathcal{B}_2 \ (-1 < r < -2/(\gamma + 1))$ $_I\mathcal{B}_1, _{III}\mathcal{B}_3,$ $_I\mathcal{B}_2 \ (-2 < s < -(1 + \gamma))$	<i>regular</i> singular type II	at ∞ : $_I\mathcal{B}_4^* \ (s < -2), _{III}\mathcal{B}_4^*,$ $_{II}\mathcal{B}_3^* \ (-4 < r < -1)$ $_I\mathcal{B}_4^* \ (-2 < s < -4(1 + \gamma)),$ $_{III}\mathcal{B}_4^*, _I\mathcal{B}_3^*, _{II}\mathcal{B}_3^* \ (r > -1)$

Table 1: Summary of our results for the case of non-interacting bulk components.

equation of state	flat brane		curved brane	
$P_2 = \gamma \rho_2$	type	balance	type	balance
$\gamma > 1$	<i>regular</i> singular big rip singular type I	${}_\nu \mathcal{B}_1$: $-8 < \nu < -6$ at ∞ ${}_\nu \mathcal{B}_1$: $\nu < -8$ ${}_\nu \mathcal{B}_2$: $\nu < -4 + 4\gamma$	<i>regular</i>	${}_\nu \mathcal{B}_3$: $\nu > -6$ at ∞
$\gamma = 1$	singular type I	${}_\nu \mathcal{B}_2$: $\nu < 0$	<i>regular</i>	${}_\nu \mathcal{B}_3$: $\nu > -6$ at ∞
$-1/2 < \gamma < 1$	singular type I	${}_\nu \mathcal{B}_2$: $\nu < -4 + 4\gamma$, ${}_\nu \mathcal{B}_1$: $\nu > -4 + 4\gamma$	<i>regular</i>	${}_\nu \mathcal{B}_3$: $\nu > -6$ at ∞
$\gamma = -1/2$	<i>regular</i> singular type II singular type I	${}_\nu \mathcal{B}_{-1/2}$: $\nu > -6$ at ∞ ${}_\nu \mathcal{B}_{-1/2}$: $\nu < -6$ ${}_\nu \mathcal{B}_1$: $-6 < \nu < 0$	<i>regular</i> singular type II	${}_\nu \mathcal{B}_{-1/2}$: $\nu > -6$ at ∞ ${}_\nu \mathcal{B}_{-1/2}$: $\nu < -6$
$-1 < \gamma < -1/2$	<i>regular</i> singular type I	${}_\nu \mathcal{B}_2$: $-4 + 4\gamma < \nu < -6$, $\nu > -6$ at ∞ ${}_\nu \mathcal{B}_1$: $\nu > -6$	singular type II	${}_\nu \mathcal{B}_3$: $\nu < -6$
$\gamma = -1$	singular type I	${}_\nu \mathcal{B}_1$: $-6 < \nu < 0$	singular type II	${}_\nu \mathcal{B}_3$: $\nu < -6$
$\gamma < -1$	singular big rip singular type I	${}_\nu \mathcal{B}_2$: $\nu > -4 + 4\gamma$ ${}_\nu \mathcal{B}_1$: $\nu > -6$	singular type II	${}_\nu \mathcal{B}_3$: $\nu < -6$

Table 2: Summary of our results for the case of interacting bulk components.

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